1. Complex and holomorphic vector bundles

**Definition 1.** Let $M$ be a differentiable manifold. A $C^\infty$ complex vector bundle consists of a family $\{E_x\}_{x \in M}$ of complex vector spaces parametrized by $M$, together with a $C^\infty$ manifold structure of $E = \bigcup_{x \in M} E_x$ such that

1. The projection map $\pi: E \to M$ taking $E_x$ to $x$ is $C^\infty$, and
2. For every $x_0 \in M$, there exists an open set $U$ in $M$ containing $x_0$ and a diffeomorphism $\phi_U: \pi^{-1}(U) \to U \times \mathbb{C}^k$ taking a vector space $E_x$ isomorphically onto $\{x\} \times \mathbb{C}^k$ for each $x \in U$; $\phi_U$ is called a trivialization of $E$ over $U$.

The dimension of the fibers $E_x$ of $E$ is called the rank of $E$; in particular, a vector bundle of rank 1 is called a line bundle. Note that for any pair of trivializations $\phi_U$ and $\phi_V$ the map $g_{UV}: U \cap V \to GL_k$ given by

$$g_{UV}(x) = (\phi_U \circ \phi_V^{-1})|_{\{x\} \times \mathbb{C}^k}$$

is $C^\infty$; the maps $G_{UV}$ are called transition functions for $E$ relative to the trivializations $\phi_U, \phi_V$. The transition functions of $E$ necessarily satisfy the identities

$$g_{UV}(x) \cdot g_{VW}(x) \cdot g_{WU}(x) = I$$

for all $x \in U \cap V \cap W$.

Conversely given an open cover $\mathcal{U} = \{U_\alpha\}$ of $M$ and $C^\infty$ maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \to GL_k$ satisfying the these identities, there is a unique complex vector bundle $E \to M$ with transition functions $\{g_{\alpha\beta}\}$: it is not hard to check that $E$ as a point set must be the union

$$\bigcup_{\alpha} U_\alpha \times \mathbb{C}^k$$

with points $(x, \lambda) \in U_\beta \times \mathbb{C}^k$ and $(x, g_{\alpha\beta}(x) \cdot \lambda) \in U_\alpha \times \mathbb{C}^k$ identified and with the manifold structure induced by the inclusions $U_\alpha \times \mathbb{C}^k \hookrightarrow E$.

As a general rule, operations on vector spaces induce operations on vector bundles. For example, if $E \to M$ is a complex vector bundle, we take the dual bundle $E^* \to M$ to be the complex vector bundle with fiber $E^*_x = (E_x)^*$; trivializations

$$\phi_U^*: E^*_U \to U \times \mathbb{C}^k$$

where $E_U = \pi^{-1}(U)$ then induce maps

$$\phi_U^*: E^*_U \to U \times \mathbb{C}^{k^*} \cong U \times \mathbb{C}^k,$$
which give $E^* = \cap E_x^*$ the structure of a manifold. The construction is most easily expressed in terms of transition functions: if $E \to M$ has transition functions $\{g_{\alpha\beta}\}$, then $E^* \to M$ is just the complex vector bundle given by transition functions $j_{\alpha\beta}(x) = t^{g_{\alpha\beta}}(x)^{-1}$.

Similarly, if $E \to M$, $F \to M$ are complex vector bundles of rank $k$ and $l$ with transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$, respectively, then one can define bundles

1. $E \oplus F$, given by transition functions
   
   $j_{\alpha\beta}(x) = \begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & h_{\alpha\beta}(x) \end{pmatrix} \in GL(\mathbb{C}^k \oplus \mathbb{C}^l)$,

2. $E \otimes F$, given by transition functions
   
   $j_{\alpha\beta}(x) = g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) \in GL(\mathbb{C}^k \otimes \mathbb{C}^l)$,

3. $\wedge^r E$, given by transition functions
   
   $j_{\alpha\beta}(x) = \wedge^r g_{\alpha\beta}(x) \in GL(\wedge^r \mathbb{C}^k)$,

In particular, $\wedge^k E$ is a line bundle $L$ given by

$\det g_{\alpha\beta}(x) \in GL(L, \mathbb{C}) = \mathbb{C}^*$,

called the determinant bundle of $E$.

OTHER WORDS: subbundle, quotient bundle

**Definition 2.** Given a $C^\infty$ map $f: M \to N$ of differentiable manifolds $M$ and $N$ and a complex vector bundle $E \to N$, we can define the pullback bundle $f^* E$ by setting

$(f^* E)_x = E_{f(x)}$.

If

$\phi: E_U \to U \times \mathbb{C}^n$

is a trivialization of $E$ in a neighborhood of $f(x)$, then the map

$f^* \phi: f^* E_{f^{-1}U} \to f^* U \times \mathbb{C}^n$

gives $f^* E$ its manifold structure over the open set $f^{-1}U$. Transition functions for the pullback $f^* E$ will, of course, be the pullback of the transition functions for $E$.

A map between vector bundles $E$ and $F$ on $M$ is given by a $C^\infty$ map $f: E \to F$ such that $f(E_x) \subset F_x$ and $f_x = f|_{E_x}: E_x \to F_x$ is linear. Note that

$Ker(f) = \cup Ker(f_x) \subset E$

and

$Im(f) = \cup Im(f_x) \subset F$

are subbundles of $E$ and $F$, respectively if and only if the maps $f_x$ all have the same rank. Two bundles $E$ and $F$ on $M$ are isomorphic if there exists a map $f: E \to F$ with $f_x: E_x \to F_x$ an isomorphism for all $x \in M$; a vector bundle on $M$ is called trivial if it is isomorphic to the product bundle $M \times \mathbb{C}^k$.

REVIEW WORDS: section, frames
**Definition 3.** A **holomorphic vector bundle** $E \to M$ is a complex vector bundle together with the structure of a complex manifold on $E$, such that for any $x \in M$ there exists $U \ni x$ in $M$ and trivialization

$$\phi_U : E_U \to U \times \mathbb{C}^k$$

that is a biholomorphic map of complex manifolds. Such a trivialization is called a **holomorphic trivialization**.

**Remark 1.** One important difference between $C^\infty$ and holomorphic vector bundles is: while there is no naturally defined exterior derivative $d$ on the space of sections of vector bundle, on a holomorphic vector bundle $E$ the $\partial$-operator $$\partial: A^{p,q}(E) \to A^{p,q+1}(E)$$ from $E$-valued $(p,q)$-forms to $E$-valued $(p,q+1)$-forms is well-defined: we take $\{e_1, \ldots, e_k\}$ any local holomorphic frame for $E$ over $U$, write $\sigma \in A^{p,q}(E)$ as

$$\sigma = \sum \omega_i \otimes e_i, \quad \omega \in A^{p,q}(U),$$

and set

$$\partial \sigma = \sum \partial \omega_i \otimes e_i.$$

**Example 1.**

1. $T^*(M) = T(M)^*$: the complex cotangent bundle,
2. $T^s(M), T^s(M)$: the holomorphic and antiholomorphic cotangent bundles,
3. $T^{s(p,q)}(M) = \bigwedge^p T^s(M) \otimes \bigwedge^q T^{s'}(M)$.

The tensor, symmetric, and exterior products of the holomorphic and complexified tangent and cotangent bundles are called **tensor bundles**.

If $V \subset M$ is a complex submanifold, we define the **normal bundle** $N_{V/M}$ to $V$ in $M$ to be the quotient of the tangent bundle to $M$, restricted to $V$ by the subbundle

$$T'(V) \hookrightarrow T'(M)|_V.$$

The **conormal** bundle $N^*_{V/M}$ to $V$ in $M$ is the dual of the normal bundle.

2. **Connections**

A smooth function with values in $\mathbb{R}^k$ on a manifold $M$ can be viewed as a section of the trivial vector bundle $M \times \mathbb{R}^k$. The theory of connections is an attempt to generalize the notion of directional derivative of (real or vector-valued) functions to sections in vector bundles.

Let $\pi: E \to M$ be a vector bundle. We are interested in operators which assign to each smooth vector field $X$ on $M$ and smooth section $\sigma$ on $E$ another smooth section of $E$ called the **covariant derivative** of $\sigma$ with respect to $X$. Of course, we would like these operators to be $\mathbb{R}$-linear, tensorial in the first variable and to satisfy the Leibniz rule. Summarizing, we have:

**Definition 4.** A **connection** $D$ on a complex vector bundle $E \to M$ is a map

$$D: A^0(E) \to A^1(E)$$

satisfying Leibnitz’ rule

$$D(f \cdot \xi) = df \otimes \xi + f \cdot D(\xi)$$

for all sections $\xi \in A^0(E)(U)$, $f \in C^\infty(U)$.
A connection is essentially a way of differentiating sections: for \( \xi \in A^0(E)(U) \) the contraction of \( D\xi \) with a tangent vector \( v \in T_x(M) \) may be thought of as the derivative of \( \xi \) in the direction \( v \). It is, however, only a first-order approximation of differentiation, inasmuch as mixed partials will in general not be equal.

Let \( e = e_1, \ldots, e_n \) be a frame for \( E \) over \( U \). Given a connection \( D \) on \( E \), we can decompose \( De_i \) into its components, writing

\[
De_i = \sum \theta_{ij} e_j.
\]

The matrix \( \theta = (\theta_{ij}) \) of 1-forms is called the *connection matrix* of \( D \) with respect to \( e \). The data \( e \) and \( \theta \) determine \( D \): for a general section \( \sigma \in A^0(E)(U) \), writing

\[
\sigma = \sum \sigma_i e_i,
\]

we have

\[
D\sigma = \sum d\sigma_i \cdot e_i + \sum \sigma_i \cdot De_i
= \sum_j (d\sigma_i + \sum_i \sigma_i \theta_{ij}) e_i.
\]

The connection matrix \( \theta \) at a point \( z_0 \in U \) depends on the choice of frame in a neighborhood of \( z_0 \): if \( e' = e'_1, \ldots, e'_n \) is another frame with

\[
e'_i(z) = \sum g_{ij}(z)e_j(z),
\]

then

\[
De'_i(z) = \sum dg_{ij} \cdot e_j + \sum g'_{ik}\theta_{kj} \cdot e_j,
\]

so that

\[
\theta'_{ij} = dg \cdot g^{-1} + g \cdot \theta_e \cdot g^{-1}.
\]

There is in general no "natural" connection on a vector bundle \( E \). If \( M \) is complex and \( E \) hermitian, however, we can make two requirements that dictate a canonical choice of connection.

1. Using the decomposition \( T^* = T^*' \oplus T^*'' \), we can write \( D = D' + D'' \), with \( D' : A^0(E) \to A^{1,0}(E) \) and \( D'' : A^0(E) \to A^{0,1}(E) \). Now we say that a connection \( D \) on \( E \) is compatible with the complex structure if \( D'' = \nabla \).

2. If \( E \) is hermitian, \( D \) is said to be compatible with the metric if

\[
d(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta).
\]

**Lemma 1.** *If \( E \) is a hermitian vector bundle, there is a unique connection \( D \) on \( E \) compatible with both the metric and the complex structure.*

**Proof.** Let \( e = e_1, \ldots, e_n \) be a holomorphic frame \( E \), and let \( h_{ij} = (e_i, e_j) \). If such a \( D \) exists, its matrix \( \theta \) with respect to \( e \) must have type \((1, 0)\), and consequently

\[
dh_{ij} = d(e_i, e_j)
= \sum_k \theta_{ik}h_{kj} + \sum_k \overline{\theta}_{jk}h_{ik}
= \text{type (1,0)} + \text{type (0,1)}.
\]
Comparing types, we have
\[ \partial h_{ij} = \sum_k \theta_{ik} h_{kj}, \quad i.e., \quad \partial h = \theta h, \]
\[ \bar{\partial} h_{ij} = \sum_k \bar{\theta}_{jk} h_{ik}, \quad \bar{\partial} h = \bar{h} \bar{\theta}, \]
and we see that \( \theta = \partial h \cdot h^{-1} \) is the unique solution to both equations. Since \( \theta \) is determined by the conditions of compatibility, \( \theta \) is well-defined globally. \( \square \)

**Definition 5.** The unique connection compatible with the complex and metric structures on \( E \) is called the associated, or metric, connection. as mentioned in the proof, its matrix with respect to a holomorphic frame is of type \((1,0)\); on the other hand if \( e_1, \ldots, e_n \) is unitary frame,
\[ 0 = d(e_i, e_j) = \theta_{ij} + \bar{\theta}_{ji}, \]
so its matrix with respect to a unitary frame is skew-hermitian.

The metric connections of hermitian vector bundles behave well with respect to bundle operations, as we see in the next two lemmas.

**Lemma 2.** Let \( E \to M \) be a hermitian vector bundle and \( F \subset E \) a holomorphic subbundle. Then \( F \) is itself a hermitian bundle with metric connection \( D_F \). On the other hand, the metric connection \( D_E \) in \( E \) and direct-sum decomposition \( E = F \oplus F^\perp \) induced by the metric give a connection \( \pi_F D_E \) in \( F \), and \( D_F = \pi_F \circ D_E \), where \( \pi_F \) is the projection onto \( F \).

**Proof.** If \( \xi \) is a section of \( F \), then \( (\pi_F \circ D_E')(\xi) = \pi_F(D''_E \xi) = \pi_F (\bar{\partial} \xi) = \bar{\partial} \xi, \) so that \( \pi_F \circ D_E \) is compatible with the complex structure. If \( \xi, \xi' \) are sections of \( F \), then
\[ d(\xi, \xi') = (D_E \xi, \xi') + (\xi, D_E \xi') \]
\[ = (\pi_F \circ D_E \xi, \xi') + (\xi, \pi_F \circ D_E \xi') \]
so that \( \pi_F \circ D_E \) is compatible with the metric. \( \square \)

Similarly, if \( E, E' \) are hermitian vector bundles, there is a natural metric on \( E \otimes E' \) given by
\[ (\lambda \otimes \lambda', \delta \otimes \delta') = (\lambda, \delta) \cdot (\lambda', \delta') \]
for \( \lambda, \delta \in E_x, \lambda', \delta' \in E'_x \). Let \( D_E, D_{E'}, D_{E \otimes E'} \) denote the metric connections on \( E, E', \) and \( E \otimes E', \) respectively, and let \( D_{E \otimes 1} \) be the connection on \( E \otimes E' \) given by
\[ (D_E \otimes 1)(\xi \otimes \zeta) = D\xi \otimes \zeta; \]
define \( 1 \otimes D_{E'} \) analogously. Then we have

**Lemma 3.**
\[ D_{E \otimes E'} = D_E \otimes 1 + 1 \otimes D_{E'}. \]

Finally, note that a hermitian metric on the holomorphic bundle \( E \) induces a metric on \( E^* \) - if \( e \) is a unitary frame for \( E, e^* \) the dual frame for \( E^* \), set
\[ (e_i^*, e_j^*) = \delta_{ij} \]
- and the metric connection \( D^* \) on \( E^* \) can be defined by the requirement
\[ d\langle \sigma, \tau \rangle = \langle D\sigma, \tau \rangle + \langle \sigma, D^*\tau \rangle \]
for $\sigma \in A^0(E)(U) \tau \in A^0(E^*)(U)$.

Now, returning to the general discussion, given a connection $D$ on a complex vector bundle $E \to M$ we can define operators $D : A^p(E) \to A^{p+1}(E)$ by forcing Leibnitz' rule

$$D(\psi \wedge \xi) = d\psi \otimes \xi + (-1)^p \psi \wedge D\xi$$

for $\psi \in A^p(U), \xi \in A^0(E)(U)$. In particular we can discuss the operator

$$D^2 : A^0(E) \to A^2(E).$$

The first fact about $D^2$ is that it is linear over $A^0$, i.e. for $\sigma$ a section of $E$ and $f$ a $C^\infty$ function,

$$D^2(f \cdot \sigma) = D(df \otimes \sigma + f \cdot D\sigma)$$

$$= -df \wedge D\sigma + df \wedge D\sigma + f \cdot D^2\sigma$$

$$= f \cdot D^2\sigma.$$

Consequently the map $D^2 : A^0(E) \to A^2(E)$ is induced by a bundle map

$$E \to \bigwedge^2 T^* \otimes \text{Hom}(E, E) = \bigwedge^2 T^* \otimes (E^* \otimes E).$$

If $e'$ is a frame for $E$, then in terms of the frame $\{E^*_i \otimes e_j\}$ for $E^* \otimes E$, we can represent $\Theta \in A^2(E^* \otimes E)$ by a matrix $\Theta_e$ of 2-forms – i.e., we can write

$$D^2 e_i = \sum \Theta_{ij} \otimes e_j;$$

$\Theta_e$ is called the curvature matrix of $D$ in terms of the frame $e$. If $\{e'_i = \sum g_{ij} e_j\}$ is another frame

$$D^2 e'_i = D^2\left(\sum g_{ij} e_j\right)$$

$$= \sum g_{ij} \Theta_{jk} e_k$$

$$= \sum g_{ij} \Theta_{jk} g^{-1}_{kl} e'_l,$$

that is,

$$\Theta_{e'} = g \cdot \Theta_e \cdot g^{-1}.$$

The curvature matrix is readily expressed in terms of the connection matrix: by definition

$$D^2 e_i = D\left(\sum \theta_{ij} \otimes e_j\right)$$

$$= \sum (d\theta_{ij} - \sum \theta_{ik} \wedge \theta_{kj}) \otimes e_j.$$ 

In matrix notation, therefore,

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e.$$

This is called the Cartan structure equation.

We can say more about $\Theta$ in the holomorphic case. If $E \to M$ is hermitian and the connection $D$ on $E$ is compatible with the complex structure, then $D'' = \bar{\partial}$ implies $D'' = 0$ and hence $\Theta^{0,2} = 0$. If, moreover, $D$ is compatible with the metric, then in terms of a unitary frame $e$, the connection matrix $\theta_e$ is skew-hermitian and hence so is $\Theta = d\theta - \theta \wedge \theta$; thus $\Theta^{2,0} = -i\Theta^{0,2} = 0$. Since the type of $\Theta$ is clearly invariant under change of frame, we
see that the curvature matrix of the metric connection on a hermitian bundle is a hermitian matrix of \((1,1)\)-forms.

To close this section, we give computations of the metric connection and curvature matrices of hermitian bundles in two special cases.

First, recall that for \(E\) a hermitian bundle with metric connection \(D\), the metric connection \(D^*\) on \(E^*\) satisfies

\[ d\langle \sigma, \tau \rangle = \langle D\sigma, \tau \rangle + \langle \sigma, D^*\tau \rangle \]

for all \(\sigma \in A^0(E)(U)\) \(\tau \in A^0(E^*)(U)\). In particular, if \(e\) is a frame for \(E\) and \(e^*\) the dual frame for \(E^*\), \(\theta\) and \(\theta^*\) the corresponding connection matrices, we have

\[ 0 = d\langle e_i, e_j^* \rangle = \theta_{ij} + \theta^*_{ji}, \]

so that \(\theta = -t^\theta\).

In view of this, a special situation holds when we consider the metric connection on the holomorphic tangent bundle of a hermitian manifold: we can compare the dual connection \(D^*\) on the holomorphic cotangent bundle with the ordinary exterior derivative. Thus

\[ D^* : A^{1,0} \to A^{1,0} \otimes A^1 = (A^{1,0} \otimes A^{1,0}) \otimes (A^{1,0} \otimes A0, 1) \]

\[ d : A^{1,0} \to A^{2,0} \oplus (A^{1,0} \otimes A^{0,1}). \]

Since \(D^*\) is compatible with the complex structure, we have \(D^{*''} = \overline{\partial}\), i.e. the two operators agree in the factor \(A^{1,0} \otimes A^{0,1}\). As will now be seen, this gives us an effective means of computing the connection matrix of \(D\). Let \(ds^2 = \sum h_{ij}dz_i \otimes d\overline{z}_j = \sum \phi_i \otimes \overline{\phi_i}\) be a hermitian metric on \(M\).

**Lemma 4.** There exists a unique matrix \(\psi_{ij}\) of 1-forms such that \(\phi + t^\psi = 0\) and

\[ d\phi_i = \sum_j \psi_{ij} \wedge \phi_j + \tau_i \]

where \(\tau_i\) is type \((2,0)\).

**Proof.** Write \(\psi = \psi' + \psi''\) for the type decomposition of \(\psi\). Then

\[ \overline{\partial}\phi_i = \sum \psi''_{ij} \wedge \phi_j \]

determines \(\psi''\), and \(\psi + t^\psi = 0\) implies \(\psi' = -t^\psi''\). (Explicitly: if we write \(\phi_i = \sum a_{ij}dz_j\), where \(\overline{a^\alpha} = h\), we have

\[ \overline{\partial}\phi_i = \sum_k a_{ik} \wedge dz_k \]

\[ = \sum_{j,k} \overline{\partial}a_{ik} \wedge a_k^{-1} \cdot \phi_j, \]

so \(\psi'' = \overline{\partial}a a^{-1} \cdot \). \(\square\)

Let \(v = v_1, \ldots, v_n\) be the frame for the tangent bundle \(T'(M)\) dual to the frame \(\phi_1, \ldots, \phi_n\); let \(\theta\) be the connection matrix of \(D\) with respect to the frame \(v\) and \(\theta^*\) the matrix for \(D^*\) in the frame \(\phi_1, \ldots, \phi_n\). Then

\[ D^{*''} = \overline{\partial} \Rightarrow \theta^{*''} = \psi'' \]

\[ \Rightarrow \theta^* = \psi \]
since $\theta^* + ^t\theta^* = 0$ and $\psi + ^t\psi = 0$. Thus we have
\[ \theta = -^t\theta^* = -^t\psi. \]

In summary, using the basic structure equation we may determine the connection matrix $\theta = -^t\psi$ in the holomorphic tangent bundle $T'(M)$ by knowing the exterior derivatives $d\phi_i$ of unitary coframe. The vector $\tau = (\tau_1, \ldots, \tau_n)$ is called the torsion; a metric is called Kähler if its torsion vanishes. Later on we shall give alternate definitions of the Kähler condition.

References