1. Differential forms

Goals

- We want to be able to integrate (holomorphic functions) on manifolds.
- Obtain a version of Stokes’ Theorem - a generalization of the fundamental theorem of calculus.
- Develop a various cohomology theories.

Remark 1. The standard definition of differential forms is essentially a real variable topic on smooth manifolds. We will follow this usual approach and also define pull-backs, cohomologies, etc. in this setting. Then via a standard trick define it on complex manifolds.

1.1. Alternating forms.

Definition 1. An \( m \)-linear function \( f \) which maps the \( m \)-fold cartesian product \( V^m \) of a vector space \( V \) into some other vector space \( W \) is called alternating if \( f(v_1, \ldots, v_m) = 0 \) whenever \( v_1, \ldots, v_m \in V \) and \( v_i = v_j \) for \( i \neq j \). We let \( \bigwedge(V^m, W) \) be the vector space of \( m \)-linear alternating functions mapping \( V^m \) into \( W \). We then define \( \bigwedge V \) by the property that if \( f \in \bigwedge(V^m, W) \) then there exists a unique \( h: \bigwedge V \to W \) with \( f(v_1, \ldots, v_m) = h(v_1 \wedge \cdots \wedge v_m) \). Then associating \( f \) with \( h \) we obtain the linear isomorphism \( \bigwedge(V^m, W) \cong \text{Hom}(\bigwedge V, W) \). We will usually think of \( W \) as \( \mathbb{R} \).

Proposition 1. A linear map \( f: V \to V' \) induces a map \( \bigwedge(V^m, W): \bigwedge(V', W) \to \bigwedge(V, W) \) defined by \( h \circ f^{(m)} \)

\[
\begin{align*}
V^m & \xrightarrow{f^{(m)}} (V')^m \\
& \xrightarrow{h} W
\end{align*}
\]

where \( f^{(m)}(v_1, \ldots, v_m) = (f(v_1), \ldots, f(v_m)) \) for \( (v_1, \ldots, v_m) \in V^m \).

Example 1. Let \( V' = V \) in the proposition with \( \text{dim}(V) = n < \infty \). Then \( (\bigwedge f)\phi = \det(f)\phi \) whenever \( \phi \in \bigwedge^n(V, W) \).

1.2. Differential forms.

Remark 2. Recall that there is a canonical isomorphism:

\[
\bigwedge^m T^*_p(M) = \bigwedge^m (\text{Hom}(T'_p(M), \mathbb{R})) \cong \text{Hom} \left( \bigwedge^m T'_p(M), \mathbb{R} \right) = \left[ \bigwedge^m T'_p(M) \right]^*
\]
Definition 2. A \textit{m-form} at \( p \in U \subset M \) is a (continuous) function \( \omega : U \to \bigwedge^m T_p^* M \). The tangent bundle is \( T'(M) = \cup_{p \in M} T'_p(M) \) and the cotangent bundle is \( T^*(M) = \cup_{p \in M} T^*_p(M) \). Then \( \bigwedge^m T^*(M) = \cup_{p \in M} \bigwedge^m T^*_p(M) \). Let \( \text{Hom}(M, \bigwedge^m T^*(M)) \) be the space of \textit{m-forms}, and requiring that the maps be differentiable, i.e. \( C^\infty \), gives us the space of \textit{differential} \( m \)-\textit{forms}

\[
\Omega^m(M) = C^\infty \left( M \to \bigwedge^m T^*(M) \right).
\]

The space of differential forms on \( M \) is \( \Omega(M) = \bigoplus_{m \in \mathbb{N}} \Omega^m(M) \), this is a graded something or other.

Definition 3 (Helpful Jargon). The set \( T'(M) = \cup_{p \in M} T'_p(M) \) is a vector bundle over \( M \), that is, to each point \( p \in M \) there is an assigned vector space \( T'_p(M) \). For every vector bundle \( E \) over \( M \) there is (by definition) a canonical projection \( \pi : E \to M \). Then a section of \( E \) is a continuous map \( s : U \to E \) defined in open sets \( U \subset M \), such that \( \pi s = \text{id}_U \). One can say a section is holomorphic, smooth, \( C^r \), etc. if the function \( s \) is. In this jargon, we define a differential \( k \)-form to be a smooth section of \( \bigwedge^k T^*_p(M) \).

Definition 4. Let \( \omega \in \Omega^{k_1}(M) \) and \( \nu \in \Omega^{k_2}(M) \). The exterior product of \( \omega \land \nu \in \Omega^{k_1+k_2}(M) \) is defined by \( (\omega \land \nu)(p) := \omega(p) \land \nu(p) \).

1.3. Pull backs.

Definition 5. Let \( f : M \to N \) be a holomorphic map, we define a \textbf{pull back} \( f^* : \Omega^k(N) \to \Omega^k(M) \) by

\[
(f^* \eta)(v_1 \land \cdots \land v_k) = \eta_{f(p)}(T'_p f \cdot v_1 \land \cdots \land T'_p f \cdot v_k),
\]

for tangent vectors \( v_1, \cdots, v_k \in T'_p(M) \) (or \( v_1 \land \cdots \land v_k \in \bigwedge^k T'_p(M) \)). Then \( f^* \) extends in the obvious way to \( f^* : \Omega^k(N) \to \Omega^k(M) \).

Proposition 2. Let \( f : M \to N \) and \( \eta_1, \eta_2 \in \Omega(N) \). Then \( f^*(\eta_1 \land \eta_2) = f^*\eta_1 \land f^*\eta_2 \).

Proposition 3. Let \( f : M \to N \) and \( g : N \to P \), then \( (f \circ g)^* = g^* \circ f^* \).

1.4. Exterior derivatives. The following Theorem can be found \cite{4} Theorem 3.1, pp22.

Theorem 1. Let \( M \) be a smooth manifold of dimension \( n \). There exists a unique \( \mathbb{R} \)-linear endomorphism \( d \) of \( \Omega(M) \), called the \textbf{exterior derivative} or \textbf{exterior differential} satisfying the following axioms:

i. \( d \) maps \( p \)-\textit{forms} to \( (p + 1) \)-\textit{forms}.
ii. \( d \) is the usual differential on functions as elements of \( \Omega^0(M) \).
iii. (Anti-derivation) If \( \eta \) is a \( p \)-\textit{form} and \( \omega \) is a \( q \)-\textit{form}, then

\[
d(\eta \land \omega) = d\eta \land \omega + (-1)^p \eta \land \omega.
\]

iv. \( d \circ d = 0 \).

v. (Naturality) If \( \psi : M \to N \) is a smooth map and \( \omega \) is a smooth \textit{form}, then \( d(\psi^* \omega) = \psi^*(d\omega) \).
Proof. As with most canonical definitions/theorems, we just define $d$ is the obvious way and check that the properties are satisfied.

Let $x_1, \ldots, x_n$ be the standard coordinates on $\mathbb{R}^n$ and take $I = \{i_1, \ldots, i_p\}$ with $i_1 < \cdots < i_p$ we denote $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then every $p$-form can be written as

$$\omega = \sum_I \omega^I dx_I,$$

as we define $d\omega$ by

$$d\omega = \sum_I d\omega^I \wedge dx_I.$$ 

□

Remark 3. By the usual differential on functions as elements of $f \in \Omega^0(M) := C^\infty(M)$, we mean

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$ 

More formally let $(U, \phi)$ be a chart. Then in $U f \circ \phi^{-1}: f(U) \subset \mathbb{R}^n \to \mathbb{R}$ is analytic (holomorphic, smooth, $C^r$, whatever) and so we interpret

$$\frac{\partial f}{\partial x_i} := \frac{\partial(f \circ \phi^{-1})}{\partial x_i}$$

2. Cohomology and the Complexification of Differential forms

2.1. de Rham cohomology.

Definition 6. A differential form $\alpha$ is closed if $d\alpha = 0$, and exact if $\alpha = d\beta$ for some form $\beta$.

Remark 4. Since $d \circ d = 0$, every exact form is closed.

Definition 7. We define the sets $Z^k(M), B^k(M) \subset \Omega^k(M)$ as

$$Z^k(M) = \{\text{closed } k\text{-forms}\}$$

$$= \ker\{d: \Omega^k(M) \to \Omega^{k+1}(M)\}$$

$$B^k(M) = \{\text{exact } k\text{-forms}\}$$

$$= \text{image}\{d: \Omega^{k-1}(M) \to \Omega^k(M)\}$$

Remark 5. Observe that the previous remark implies that $B^k(M) \subset Z^k(M)$.

Remark 6. Georges de Rham 1903-1990

Definition 8. The $k$th de Rham cohomology group (vector space even) is given as the quotient group

$$H^k_{DR}(M) = \frac{Z^k(M)}{B^k(M)};$$

that is, we look at closed $k$-forms and identify any whose difference is an exact $k$-form.

Let $\alpha \in Z^k(M) \subset \Omega^k(M)$ then the equivalence class that contains $\alpha$ is denoted $[\alpha]$ and called the cohomology class of $\alpha$. 
Proposition 4. If \( f : M \to N \) is smooth and \( [\beta] \in H^k(N) \) then the pull back \( f^* \) defines a linear map (group homomorphism, v.s. homom.)
\[
f^* : H^k(N) \to H^k(M)
\]
by \( f^*([\beta]) = [f^*\beta] \).

Proof. Suppose \( [\beta] \in H^k(N) \). If \( \beta \) is closed, then \( f^*\beta \) is closed. Therefore \( [f^*\beta] \) is a cohomology class in \( H^k(M) \).

Now it needs to be seen that \( [f^*\beta] \) depends only on the cohomology class of \( \beta \). To see this, let \( \beta - \beta' \in d\eta \in B^k(N) \), and so \( f^*\beta - f^*\beta' = df^*\eta \in B^k(M) \). \( \square \)

Remark 7 (Poincaré’s Lemma). On a contractable domain every closed form is exact, that is, \( d\alpha = 0 \) if and only if \( \alpha = d\beta \) for some \( \beta \). Therefore the de Rham cohomology is locally trivial. However this is also one of the more interesting properties of this cohomology as it allows one to obtain topological information using purely differential methods. Since at this moment we are primarily interested in complex geometry, I’ll have to save this interesting result for coffee time.

2.2. Return to Griffiths and Harris with a change of notation. In \([2]\), they use \( A^p(M, \mathbb{R}) := \Omega^p(M) \) to denote the set of differential \( p \)-forms on \( M \), \( Z^p(M, \mathbb{R}) = Z^p(M) \) the closed \( p \)-forms, and \( d(A^{p-1}(M, \mathbb{R})) := B^p(M) \) the exact \( p \)-forms.

Then the quotient group \( H^p_{DR}(M, \mathbb{R}) = Z^p(M, \mathbb{R})/d(A^{p-1}(M, \mathbb{R})) \) form the de Rham cohomology groups of \( M \).

Let \( A^p(M) := A^p(M, \mathbb{C}) \) to denote the set of complex differential \( p \)-forms on \( M \), \( Z^p(M) := Z^p(M, \mathbb{C}) \) the closed \( p \)-forms, and \( dA^{p-1}(M) := d(A^{p-1}(M, \mathbb{C})) \) the exact \( p \)-forms.

Then the quotient group \( H^p_{DR}(M) := Z^p(M)/dA^{p-1}(M) \) form the de Rham cohomology groups of \( M \).

Then \( H^p_{DR}(M) = H^p_{DR}(M, \mathbb{R}) \otimes \mathbb{C} \).

Recall that \( T_{\mathbb{C},p}(M) = T_p^s(M) \oplus T_p^u(M) \). Therefore by taking duals
\[
T_{\mathbb{C},p}^*(M) = T_p^s(M) \oplus T_p^u(M)
\]
\[
\Rightarrow \bigwedge^m T_{\mathbb{C},p}^*(M) = \bigoplus_{i+j=m} \left( \bigwedge^i T_p^s(M) \oplus \bigwedge^j T_p^u(M) \right)
\]
\[
\Rightarrow A^m(M) = \bigoplus_{i+j=m} A^{i,j}(M)
\]

Definition 9. A form \( \phi \in A^{p,q}(M) \) is said to be of type or bidegree \((p,q)\). By the way of notation, we denote by \( \pi^{(p,q)} \) the projection maps
\[
A^*(M) \to A^{p,q}(M),
\]
so that for \( \phi \in A^*(M) \),
\[
\phi = \sum \pi^{(p,q)} \phi;
\]
we usually write \( \phi^{(p,q)} \) for \( \pi^{(p,q)} \phi \).

If \( \phi \in A^{p,q}(M) \), then
\[
d\phi \in A^{p+1,q}(M) \oplus A^{p,q+1}(M).
\]
We can then define two maps:

\[ \partial : A^{p,q}(M) \to A^{p+1,q}(M) \]
\[ \overline{\partial} : A^{p,q}(M) \to A^{p,q+1}(M) \]

by

\[ \partial = \pi^{(p+1,q)} \circ d, \quad \overline{\partial} = \pi^{(p,q+1)} \circ d, \]

and so

\[ d = \partial + \overline{\partial}. \]

In local coordinates \( z = (z_1, \ldots, z_n) \), a form \( \phi \in A^m(M) \) is of type \((p, q)\) if we can write

\[ \phi(z) = \sum_{|I|=p, |J|=q} \phi_{IJ}(z) \, dz_I \wedge d\bar{z}_J, \]

where for each multiindex \( I = \{i_1, \ldots, i_p\} \),

\[ dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}. \]

The operators \( \partial \) and \( \overline{\partial} \) are then given by

\[ \partial \phi(z) = \sum_{I,J,j} \frac{\partial}{\partial z_j} \phi_{IJ}(z) \, dz_j \wedge dz_I \wedge d\bar{z}_J, \]
\[ \overline{\partial} \phi(z) = \sum_{I,J,j} \frac{\partial}{\partial \bar{z}_j} \phi_{IJ}(z) \, dz_j \wedge dz_I \wedge d\bar{z}_J. \]

In particular, we say that a form \( \phi \) of type \((p, 0)\) is holomorphic if \( \overline{\partial} \phi = 0 \); which is the case if and only if

\[ \phi(z) = \sum_{|I|=p} \phi_I(z) \, dz_I \]

with \( \phi_I(z) \) holomorphic.

**Proposition 5.** Let \( f : M \to N \) be a holomorphic map of complex manifolds,

\[ f^*(A^{p,q}(N)) \subset A^{p,q}(M) \]

and \( \overline{\partial} \circ f^* = f^* \circ \overline{\partial} \) on \( A^{p,q}(N) \).

**Definition 10.** Let \( Z_{\overline{\partial}}^{p,q}(M) \) denote the space of \( \overline{\partial} \)-closed forms of the type \((p, q)\).

**Proposition 6.** \( \partial^2 = 0, \overline{\partial}^2 = 0 \) and \( \partial \overline{\partial} = -\overline{\partial} \partial \).

**Proof.** Let \( \omega \in A^{p,q}(M) \). Then

\[ 0 = d^2 \omega = (\partial + \overline{\partial})^2 \omega = \partial^2 \omega + (\partial \overline{\partial} + \overline{\partial} \partial) \omega + \overline{\partial}^2 \omega. \]

However \( \partial^2 \omega \in A^{p+2,q}(M) \), \( (\partial \overline{\partial} + \overline{\partial} \partial) \omega \in A^{p+1,q+1}(M) \) and \( \overline{\partial}^2 \omega \in A^{p,q+2}(M) \), and are therefore of three distinct types. If they are to sum to zero, then each must be zero. \( \square \)
Definition 11. The result of the last proposition is that $\dbar$ forms a cohomology theory, called the Dolbeault cohomology, that is, the $(p,q)^{th}$ Dolbeault cohomology group is defined as

$$H^{p,q}_{\dbar}(M) = \frac{Z^{p,q}_{\dbar}(M)}{\dbar(A^{p,q-1}(M))}.$$ 

Corollary 1. If $f : M \to N$ is a holomorphic map of complex manifolds, then $f$ induces a map

$$f^* : H^{p,q}_{\dbar}(N) \to H^{p,q}_{\dbar}(M).$$

Theorem 2 ($\dbar$-Poincaré Lemma). Let $\Delta$ be a polycylinder in $\mathbb{C}^n$, that is, $\Delta = \Omega_1 \times \cdots \times \Omega_n$ with each $\Omega_j$ bounded, open and non-empty in $\mathbb{C}$, then

$$H^{p,q}_{\dbar}(\Delta) = 0$$
for all $q \geq 1$.

3. Calculus on Manifolds

Definition 12. Formally a Riemannian metric on $M$ is a section of the of the vector bundle $S^2 T^* (M)$, that is, the (positive) symmetric bilinear forms on $T(M)$. That is a function $p \mapsto g_p(\cdot, \cdot) =: \langle \cdot, \cdot \rangle_p$ where $g_p : T_p(M) \times T_p(M) \to \mathbb{R}$.

A Hermitian metric on $M$ will be a Riemannian metric on $M$ which is compatible with the complex structure of $M$. So for each $p \in M$ we have a (positive) definite Hermitian inner product

$$\langle \cdot, \cdot \rangle_p : T_{\mathbb{C},p}(M) \times T_{\mathbb{C},p}(M) \to \mathbb{C}$$
on $T_{\mathbb{C},p}(M) = T'_p(M) \oplus T''_p(M)$.

The compatibility condition is

$$\langle \alpha, \beta \rangle = \langle i\alpha, i\beta \rangle,$$
where $\alpha, \beta \in T_{\mathbb{C},p}(M)$. More generally for any almost complex structure $J$ on $M$, the compatibility condition is

$$\langle \alpha, \beta \rangle = \langle J\alpha, J\beta \rangle.$$

A compatible Riemann metric on $M$ will be symmetric and satisfy the following three conditions:

$$\langle \overline{\alpha}, \overline{\beta} \rangle = \overline{\langle \alpha, \beta \rangle};$$

$$\langle \alpha, \beta \rangle = 0$$
for $\alpha, \beta$ in $T'_p(M)$;

$$\langle \overline{\alpha}, \alpha \rangle > 0$$
unless $\alpha = 0$ (if positive).

Vice versa, a symmetric complex bilinear form $\langle \cdot, \cdot \rangle$ on $T_{\mathbb{C},p}(M)$ satisfying these three conditions is the complex bilinear extension of a compatible (positive) Riemannian metric. That is to say if $h$ is a Hermitian metric and $g$ is a Riemannian metric we have the condition

$$h(x, y) = g(x, y) + g(Jx, Jy),$$
for any almost complex structure $J$ on $M$.

In [2] a Hermitian metric on $M$ is usually denoted $ds^2$. The real part of a Hermitian metric is a Riemannian metric, i.e.

$$\text{Re} \ ds^2 : T_{\mathbb{R},p}(M) \times T_{\mathbb{R},p}(M) \to \mathbb{R}.$$
and often called the *induced Riemannian metric* on $M$. When we speak of distance, area, or volume on a complex manifold with Hermitian metric, we always refer to the induced Riemannian metric.

We also may observe that $\text{Im } ds^2: T_{\mathbb{R},p}(M) \times T_{\mathbb{R},p}(M) \to \mathbb{R}$ is alternating, and so it represents a real differential form of degree 2; $\omega = -\frac{1}{2} \text{Im } ds^2$ is called the *associated $(1,1)$-form* of the metric. When $\omega$ is closed, $\omega$ is a Kähler form (to be defined later).

A real differential $(1,1)$-form $\omega$ is called a *positive $(1,1)$-form* if for every $p \in M$ and $\nu \in T'_p(M)$ we have

$$\sqrt{-1} \langle \omega(p), \nu \wedge \overline{\nu} \rangle > 0.$$ 

In local coordinates $z$ on $M$ a form is positive if

$$\omega(z) = \frac{-1}{2} \sum_{i,j} h_{i,j}(z) dz_i \wedge d\overline{z}_j,$$

with $H(z) = (h_{i,j}(z))_{i,j}$ a positive definite Hermitian matrix for every $z$.

A *coframe* for the Hermitian metric is an $n$-tuple of $(1,0)$-forms $(\phi_1, \ldots, \phi_n)$ such that

$$ds^2 = \sum_i \phi_i \otimes \overline{\phi}_i.$$ 

**Proposition 7.** Let $f: M \to N$ be a holomorphic map such that

$$f_* := T_p f: T'_p(M) \to T'_p(N)$$

is injective, then a Hermitian metric on $N$ induces a Hermitian metric on $M$. Furthermore the pull-back of the associated $(1,1)$-form of the metric on $N$ is associated $(1,1)$-form of the induced metric on $M$. Note that $[2]$ denotes $T_p f$ by $f_*$ and that the pull-back goes the opposite direction from $f$, i.e.

$$f^*: A^{1,1}(N) \to A^{1,1}(M)$$

**Example 2.** The Hermitian metric on $\mathbb{C}^n$ given by

$$ds^2 = \sum_{i=1}^n dz_i \otimes d\overline{z}_i$$

is called the *Euclidean* or *standard metric* on $\mathbb{C}^n = \mathbb{R}^{2n}$.

**Example 3.** Let $Z_0, \ldots, Z_n$ be coordinates in $\mathbb{C}^{n+1}$ and denote by $\pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the standard projection map. Let $U \subset \mathbb{P}^n$ be an open set and $Z: U \to \mathbb{C}^{n+1} \setminus \{0\}$ a lifting of $U$, i.e. a holomorphic map with $\pi \circ Z = id$; consider the differential form

$$\omega = \frac{-1}{2\pi} \partial \overline{\partial} \log ||Z||^2.$$
If \( Z': U \to \mathbb{C}^{n+1} \setminus \{0\} \) is another lifting, then \( Z' = f \cdot Z \) with \( f \) a nonzero holomorphic function, so that
\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \|Z'\|^2 = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log (\|Z\|^2 + \log f + \log \bar{f})
\]
\[
= \omega + \frac{\sqrt{-1}}{2\pi} (\partial \bar{\partial} \log f - \bar{\partial} \partial \log \bar{f})
\]
\[
= \omega
\]
Therefore \( \omega \) is independent of the lifting chosen; since liftings always exist locally, \( \omega \) is a globally defined differential form in \( \mathbb{P}^n \). Clearly \( \omega \) is of type \((1,1)\). To see that \( \omega \) is positive, first note that the unitary group \( U(n+1) \) acts transitively \((\ast)\) on \( \mathbb{P}^n \) and leaves the form \( \omega \) invariant, so that \( \omega \) is positive everywhere if it is positive at one point. Now let \( \{w_i = Z_i/Z_0\} \) be coordinates on the open set \( U_0 = (Z_0 \neq 0) \) in \( \mathbb{P}^n \) and use the lifting \( Z = (1,w_1,\ldots,w_n) \) on \( U_0 \); we have
\[
\omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log (1 + \sum w_i \bar{w}_i)
\]
\[
= \frac{\sqrt{-1}}{2\pi} \partial \left( \sum \frac{w_i \bar{w}_i}{1 + \sum w_i \bar{w}_i} \right)
\]
\[
= \frac{\sqrt{-1}}{2\pi} \left( \sum \frac{d\bar{w}_i \wedge d\bar{w}_i}{1 + \sum w_i \bar{w}_i} - \frac{\left( \sum \bar{w}_i d\bar{w}_i \right) \wedge \left( \sum w_i d\bar{w}_i \right)}{(1 + \sum w_i \bar{w}_i)^2} \right).
\]
At the point \([1,0,\ldots,0]\),
\[
\omega = \frac{\sqrt{-1}}{2\pi} \sum d\bar{w}_i \wedge d\bar{w}_i > 0.
\]
Thus \( \omega \) defines a Hermitian metric on \( \mathbb{P}^n \), called the Fubini-Study metric.

**Remark 8.** \((\ast)\) The unitary group \( U(n+1) \) is the set of all \( n+1 \times n+1 \) matrices \( U \) such that \( UU^* = I_{n+1} \). A group \( G \) acts transitively on a set \( X \) if for any two \( x,y \in X \) there is a \( g \in G \) such that \( gx = y \).

**Theorem 3** (Wirtinger’s Theorem).

\[
\text{vol}(S) = \frac{1}{d!} \int_S \omega^d.
\]

**Proof.** The interplay between the real and imaginary parts of a Hermitian metric now gives us the Wirtinger theorem, which expresses another fundamental difference between Riemannian and Hermitian differential geometry. Let \( M \) be a complex manifold, \( z = (z_1,\ldots,z_n) \) local coordinates on \( M \), and
\[
ds^2 = \sum \phi_i \otimes \bar{\phi}_i
\]
a Hermitian metric on \( M \) with associated \((1,1)\)-form \( \omega \). Write \( \phi_i = \alpha_i + \sqrt{-1} \beta_i \); then the associated Riemannian metric on \( M \) is
\[
\text{Re } ds^2 = \sum \alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i
\]
and the volume associated to \( \text{Re} \, ds^2 \) is given by
\[
d\mu = \alpha_1 \land \beta_1 \land \cdots \land \alpha_n \land \beta_n.
\]

On the other hand we have
\[
\omega = \sum \alpha_i \land \beta_i
\]
so that the \( n \)th exterior power
\[
\omega^n = n! \cdot \alpha_1 \land \beta_1 \land \cdots \land \alpha_n \land \beta_n
\]
\[
= n! \cdot d\mu
\]

Now let \( S \subset M \) be a complex submanifold of dimension \( d \). The \((1,1)\)-form associated to the metric induced on \( S \) by \( ds^2 \) is just \( \omega|_S \) and apply the above to the induced metric on \( S \). \( \square \)

**Remark 9.** In particular this shows us that
\[
\int_E \omega = 0
\]
when \( \text{dim}(E) < \text{deg}(\omega) \). Furthermore with a slight abuse of notation this can be stated as \( A^{p,q}(E) = 0 \) when either \( p \) or \( q \) is greater than the dimension of \( E \). Here \( E \) is any set where the integration is defined, i.e. manifolds, submanifolds, analytic varieties, subvarieties.

**Remark 10.** The fact that the volume of a complex submanifold \( S \) of the complex manifold \( M \) is expressed as the integral over \( S \) of a globally defined differential form on \( M \) is quite different from the real case. For a \( C^\infty \) arc
\[
t \mapsto (x(t), y(t))
\]
in \( \mathbb{R}^n \) the element of arc length is given by
\[
\sqrt{x'(t)^2 + y'(t)^2} \, dt
\]
which is not, in general, the pullback of any differential form in \( \mathbb{R}^2 \). ( The coefficient of \( dt \) is not \( C^\infty \). )

We now get the manifold version of the fundamental theorem of calculus
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

**Theorem 4 (Stokes’ Theorem).** Let \( K \) be a compact subset of \( M \) with piecewise \( C^1 \) boundary, and \( u \) any differential form of class \( C^1 \) with degree \( m - 1 \) on \( M \). We then have
\[
\int_{\partial K} u = \int_K du.
\]

**Theorem 5 (Stokes’ Theorem for Analytic Varieties).** For \( M \) a complex manifold and \( V \subset M \) an analytic subvariety of dimension \( k \), and \( \phi \) a differential form of degree \( 2k - 1 \) with compact support in \( M \),
\[
\int_V d\phi = 0.
\]
Proof. I’m just going to point out the main idea. First observe that if $\phi$ has degree $2k - 1$ then its either $p$ or $q$ is $\geq k$ where $(p, q)$ is the bidegree of $\phi$. We now apply the usual version of Stokes’ Theorem (a little work is needed to make it apply in this case)

$$\int_V d\phi = \int_{\partial V} \phi.$$ 

However $\dim(\partial V) = k - 1$, and so by the remark following Wirtinger’s Theorem, we have that this right integral must be 0.

Remark 11. Technically we haven’t defined integration over an analytic variety yet. Recall that the smooth part of the variety $V^* = V \setminus V_s$ is a submanifold. Therefore one defines the integration over $V$ by integration over $V^*$.

References