1. Definition of a Complex Manifold

Let $M$ be a paracompact Hausdorff space (generally assumed to be connected).

**Definition 1.** We say that $M$ is a complex manifold if there exists an open covering $\{U_\alpha\}$ of $M$ and coordinate maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\phi_\alpha \circ \phi_\beta^{-1}$ is holomorphic on $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all $\alpha$ and $\beta$. We call $\phi_\alpha(z) = (z_1^\alpha, z_2^\alpha, \ldots, z_n^\alpha)$ the local coordinates for the point $z$.

**Related Words:** charts, atlas

**Example 1.** A one-dimensional complex manifold is called a Riemann surface.

**Example 2.** Let $\mathbb{P}^n$ denote the set of lines through the origin in $\mathbb{C}^{n+1}$. A line $l \subset \mathbb{C}^{n+1}$ is determined by any $Z \neq 0 \in l$, so we can write $\mathbb{P}^n = \{[Z] \neq 0 \in \mathbb{C}^{n+1}\}$. We take $U_i = \{[Z] : Z_i \neq 0\} \subset \mathbb{P}^n$ of lines not contained in the hyperplane ($Z_i = 0$), there is a bijective map $\phi_i$ to $\mathbb{C}^n$ given by

$$\phi_i([Z_0, Z_1, \ldots, Z_n]) = \left(\frac{Z_0}{Z_i}, \frac{Z_1}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right).$$

On $z_j \neq 0 = \phi_i(U_j \cap U_i) \subset \mathbb{C}^n$, $\phi_j \circ \phi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \frac{z_2}{z_j}, \ldots, \frac{z_n}{z_j}\right)$ is holomorphic. Thus $\mathbb{P}^n$ has the structure of a complex manifold, called complex projective space. The “coordinates” $Z + [Z_0, \ldots, Z_n]$ are called homogeneous coordinates on $\mathbb{P}^n$. $\mathbb{P}^n$ is compact, since we have a continuous surjective map from the unit sphere in $\mathbb{C}^{n+1}$ to $\mathbb{P}^n$. Note that $\mathbb{P}^1$ is just the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Any inclusion $\mathbb{C}^{k+1} \to \mathbb{C}^{n+1}$ induces an inclusion $\mathbb{P}^k \to \mathbb{P}^n$; the image of such a map is called a linear subspace of $\mathbb{P}^n$. The image of the hyperplane in $\mathbb{C}^{n+1}$ is again called a hyperplane, the image of 2-plane $\mathbb{C}^2 \subset \mathbb{C}^{n+1}$ is a line, and in general the image of a $\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}$ is called a $k$-plane. We may speak of linear relations among points in $\mathbb{P}^n$ in these terms; for example, the span of a collection $\{p_i\}$ of points in $\mathbb{P}^n$ is taken to be the image in $\mathbb{P}^n$ of the subspace in $\mathbb{C}^{n+1}$ spanned by the lines $\pi^{-1}(p_i)$; $k$ points are said to be linearly independent if their corresponding lines in $\mathbb{C}^{n+1}$ are, that is, if their span in $\mathbb{P}^n$ is a $(k-1)$-plane.

*Date:* February 12, 2010.
Note that the set of hyperplanes in \( \mathbb{P}^n \) corresponds to the set \( \mathbb{C}^{n+1} - \{0\} \) of nonzero linear functionals on \( \mathbb{C}^{n+1} \) modulo scalar multiplication; it is thus itself a projective space, called the dual projective space and denoted \( \mathbb{P}^{n*} \).

It is sometimes convenient to picture \( \mathbb{P}^n \) as the compactification of \( \mathbb{C}^n \) obtained by adding on the hyperplane \( H \) at infinity. In coordinates the inclusion \( \mathbb{C}^n \to \mathbb{P}^n \) is \((z_1,\ldots,z_n) \to [1,z_1,\ldots,z_n]\); \( H \) has equation \((z_0 = 0)\), and the identification \( H \cong \mathbb{P}^{n-1} \) comes by considering the hyperplane at infinity as the directions in which we can go to infinity in \( \mathbb{C}^n \).

**Example 3.** Let \( \Lambda = \mathbb{Z}^k \subset \mathbb{C}^n \) be a discrete lattice. Then the quotient group \( \mathbb{C}^n / \Lambda \) has a structure of a complex manifold induced by the projection map \( \pi: \mathbb{C}^n \to \mathbb{C}^n / \Lambda \). It is compact if and only if \( k = 2n \); in this case \( \mathbb{C}^n / \Lambda \) is called a complex torus.

### 2. Holomorphic Functions

**Definition 2.** A function \( f \) on \( U \subset M \) is holomorphic if \( f \circ \phi^{-1}_\alpha \) is holomorphic on \( \phi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n \) for all \( \alpha \). We use the notation \( \mathcal{O}(M) \) for the set of holomorphic functions on \( M \).

**Example 4.** Let \( \pi: N \to M \) is a topological covering space and \( M \) is a complex manifold, the \( \pi \) gives \( N \) the structure of a complex manifold; if \( N \) is a complex manifold and the automorphisms of \( N \) are holomorphic, then \( M \) inherits a complex manifold structure from \( N \).

**Example 5.** The Hopf surface is defined to be the quotient of \( \mathbb{C}^2 \setminus \{0\} \) by the group of automorphisms generated by \( z \mapsto 2z \). The Hopf surface is simplest example of a compact complex manifold that cannot be imbedded in projective space of any dimension.

**Definition 3 (Proposition).** Let \( M \) and \( N \) be complex manifolds with \( f : M \to N \) be a map between them and suppose \((U_\alpha,x_\alpha)\) and \((V_\beta,y_\beta)\) are atlases for \( M \) and \( N \) respectively. Then \( f \) is holomorphic if \( y_\beta \circ f \circ x_\alpha^{-1} \) is a holomorphic function on \( x_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \subset \mathbb{C}^n \) for all \( \alpha \) and \( \beta \).

### 3. Holomorphic Tangent Space

RELATED WORDS: real tangent space \( T_{\mathbb{R},p}(M) \), complexified tangent space \( T_{\mathbb{C},p}(M) = T_{\mathbb{R},p}(M) \otimes \mathbb{C} \), antiholomorphic tangent space \( T''_p(M) = \overline{T'_p(M)} \)

**Definition 4.** The set of derivations at \( p \in M \), that is linear maps \( D : \mathcal{O}(M) \to \mathbb{C} \) such that

\[
D(fg) = D(f) \cdot g(p) + f(p) \cdot D(g),
\]

forms a complex \( n \) dimensional vector space, denoted \( T'_p(M) \).

Alternate definition (1):

**Definition 5.** Let \( p \in U_\alpha \). Let \( \gamma_1, \gamma_2 : \mathbb{D} \to M \) be two analytic arcs (holomorphic curves) with \( \gamma_1(0) = \gamma_2(0) = p \). We call \( \gamma_1 \) and \( \gamma_2 \) tangent at \( 0 \) if \( \frac{\partial \gamma_1}{\partial z}(0) = \frac{\partial \gamma_2}{\partial z}(0) \). This defines an equivalence relation on such curves, and the equivalence classes are known as tangent vectors of \( M \) at \( z \), denoted \( T'_p(M) \).

Alternate definition (2):
Definition 6. Let \( p \in U_a \). Let \( \Delta \) be the some polydisc in \( \mathbb{C}^n \), and \( \gamma : \Delta \rightarrow U \subset M \). Then \( T'_p(M) \) is the union of the tangent lines at \( p \) over all such curves \( \gamma \).

Remark 1.

\[ T_{C,p}(M) = T'_p(M) \oplus T''_p(M) \]

Proposition 1. Let \( f : M \rightarrow N \) holomorphic with \( p \in M \) and \( f(p) \in N \), then there is a map \( T'_p f : T'_p(M) \rightarrow T'_{f(p)}(N) \).

proof using derivations. Let \( v_p \in T'_p(M) \). Then define \( T'_p f \cdot v_p \) as a derivation by \( (T'_p f \cdot v_p)g = v_p(g \circ f) \). Then we just check the properties. Let \( g, h \in \mathcal{O}(N) \), and so \( g \circ f, h \circ f \in \mathcal{O}(M) \). Therefore

\[ (T'_p f \cdot v_p)(g \cdot h) = v_p((g \circ f) \cdot (h \circ f)) \]

\[ = v_p(g \circ f)(h \circ f)(p) + (g \circ f)(p)v_p(h \circ f) \]

\[ = (T'_p f)(g)(h \circ f(p)) + g(f(p))(T''_p f)(h) \]

So \( T'_p f \) is a derivation at \( f(p) \) on \( N \). \( \square \)

proof using holomorphic curves. Suppose \( v_p \in T'_p(M) \), find a representative holomorphic curve \( \gamma \) such that \( \gamma(0) = p \) with \( v_p = [\gamma] \), then we define \( T'_p f(v_p) = [f \circ \gamma] \in T'_{f(p)}(N) \). \( \square \)

Definition 7. Let \( M \) and \( N \) be complex manifolds, \( z = (z_1, \ldots, z_n) \) be holomorphic coordinates centered at \( p \in M \), \( w = (w_1, \ldots, w_m) \) holomorphic coordinates at \( q \in N \), and \( F : M \rightarrow N \) a holomorphic map with \( f(p) = q \). We have different Jacobians of \( f \) depending on which tangent spaces we are playing with. For the holomorphic tangent space we have:

\[ \mathcal{J}(f) = \mathcal{J}(w \circ f \circ z^{-1}) = \left( \frac{\partial w_\alpha}{\partial z_j} \right), \]

where \( w \circ f \circ z^{-1} : U \rightarrow V \) with \( U, V \subset \mathbb{C}^n \).

Remark 2.

\[ J_C(f) = \begin{pmatrix} \mathcal{J}(f) & 0 \\ 0 & \mathcal{J}(f) \end{pmatrix} \]

Remark 3. Any complex manifold has a natural orientation which is preserved under holomorphic maps.

4. Submanifolds and Subvarieties

Theorem 1 (Inverse Function Theorem). Let \( U, V \) be open sets in \( \mathbb{C}^n \) with \( 0 \in U \) and \( f : U \rightarrow V \) a holomorphic map with \( \mathcal{J}(f) = (\partial f_i/\partial z_j) \) nonsingular at \( 0 \). Then \( f \) is one-to-one in a neighborhood of \( 0 \), and \( f^{-1} \) is holomorphic at \( f(0) \).

Theorem 2 (Implicit Function Theorem). Given \( f_1, \ldots, f_k \in \mathcal{O}_k = \mathcal{O}(\mathbb{C}^k) \) with

\[ \text{det} \left( \frac{\partial f_i}{\partial z_j}(0) \right)_{1 \leq i, j \leq k} \neq 0, \]

there exists functions \( w_1, \ldots, w_k \in \mathcal{O}_{n-k} \) such that in a neighborhood of \( 0 \) in \( \mathbb{C}^n \),

\[ f_1(z) = \cdots = f_k(z) = 0 \Leftrightarrow z_i = w_i(z_{k+1}, \ldots, z_n) \quad 1 \leq i \leq k. \]
Proposition 2. If \( f: U \to V \) is a one-to-one holomorphic map of open sets in \( \mathbb{C}^n \) then \(|\mathcal{J}(f)| \neq 0\), i.e. \( f^{-1} \) is holomorphic.

Proof. We prove this by induction on \( n = 1 \); the case \( n = 1 \) is clear. Let \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \) be coordinates on \( U \) and \( V \), respectively, and suppose that \( \mathcal{J}(f) \) has rank \( k \) at \( 0 \in U \); we may assume then that the matrix \( ((\partial f_i/\partial z_j)(0))_{0 \leq i,j \leq k} \) is nonsingular. Set

\[
\begin{align*}
z'_i &= f_i(z), & 1 \leq i \leq k, \\
\zeta'_\alpha &= z_\alpha, & k + 1 \leq \alpha \leq n;
\end{align*}
\]

by the inverse function theorem, \( z' = (z'_1, \ldots, z'_n) \) is a holomorphic coordinate system for \( U \) near \( 0 \). But now \( f \) maps the locus \( (z'_1 = \cdots = z'_k = 0) \) one-to-one onto the locus \( (w_1 = \cdots = w_k = 0) \) and the Jacobian \((\partial f_\alpha/\partial z'_\beta)\) of \( f|_{(z'_1=\cdots=z'_k=0)} \) is singular at \( 0 \); by the induction hypothesis, either \( k = 0 \) or \( k = n \). We see that the Jacobian matrix of \( f \) vanishes identically wherever its determinant is zero, i.e. that \( f \) maps every connected component of the locus \(|\mathcal{J}(f)| = 0\) to a single point in \( V \). Since \( f \) is one-to-one and the zero locus of the holomorphic function \(|\mathcal{J}(f)|\) is positive dimensional if nonempty, it follows that \(|\mathcal{J}(f)| \neq 0\). \( \square \)

Remark 4. Note that this proposition is not true in the real case, where the map \( t \mapsto t^3 \) on \( \mathbb{R} \) is one-to-one but does not have a \( C^\infty \) inverse.

5. Submanifolds and analytic varieties

Definition 8. A complex submanifold \( S \) of a complex manifold \( M \) is a subset \( S \subset M \) given locally either as the zeros of a collection \( f_1, \ldots, f_k \) of holomorphic functions with rank \( \mathcal{J}(f) = k \), or as the image of an open set \( U \) in \( \mathbb{C}^{n-k} \) under a map \( f: U \to M \) with rank \( \mathcal{J}(f) = n-k \).

Remark 5. These definitions are the same by the implicit function theorem. The submanifold is a complex manifold of dimension \( n-k \).

Definition 9. We say that a subset \( V \) of an open set \( U \subset \mathbb{C}^n \) is an analytic variety in \( U \) if, for any \( p \in U \), there exists a neighborhood \( U' \) of \( p \) in \( U \) such that \( V \cap U' \) is the common zero locus of a finite collection of holomorphic function \( f_1, \ldots, f_k \) on \( U' \). In particular, \( V \) is called an analytic hypersurface if \( V \) is locally the zero locus of a single nonzero holomorphic function \( f \).

An analytic variety \( V \subset U \subset \mathbb{C}^n \) is said to be irreducible if \( V \) cannot be written as the union of two analytic varieties \( V_1, V_2 \subset U \) with \( V_1, V_2 \neq V \); it is said to be irreducible at \( p \in V \) if \( V \cap U' \) irreducible for small neighborhoods \( U' \) of \( p \) in \( U \). Note first that if \( f \in \mathcal{O}_n \) is irreducible in the ring \( \mathcal{O}_n \) then the analytic hypersurface \( V = \{f(z) = 0\} \) given by \( f \) in a neighborhood of \( 0 \) is irreducible at \( 0 \): if \( V = V_1 \cup V_2 \), with \( V_1, V_2 \) analytic varieties \( \neq V \), then there exists \( f_1, f_2 \in \mathcal{O}_n \) with \( f_1 \) (respectively \( f_2 \)) vanishing identically on \( V_1 \) (respectively \( V_2 \)) but not on \( V_2 \) (respectively \( V_1 \)). By the Nullstellensatz \( f \) must divide the product \( f_1 \cdot f_2 \), i.e. either \( V_1 \supset V \) or \( V_2 \supset V \), a contradiction.

Suppose \( V \subset U \subset \mathbb{C}^n \) is an analytic hypersurface, given by \( V = \{f(z) = 0\} \) in a neighborhood of \( 0 \in V \). Since \( \mathcal{O}_n \) is a UFD, we can write \( f = f_1 \cdots f_n \) with \( f_i \) irreducible in \( \mathcal{O}_n \);
if we set \( V_i = \{ f_i(z) = 0 \} \) then we have
\[
V = V_1 \cup \cdots \cup V_n
\]
with \( V_i \) irreducible at 0. Thus if \( p \) is any point on any analytic hypersurface \( V \subset U \subset \mathbb{C}^n \), \( V \) can be expressed uniquely in some neighborhood \( U' \) of \( p \) as the union of a finite number of analytic hypersurfaces irreducible at \( p \).

**Definition 10.** An analytic subvariety \( V \) of a complex manifold \( M \) is a subset given locally as the zeros of a finite collection of holomorphic functions. A point \( p \in V \) is called a smooth point of \( V \) if \( V \) is a submanifold of \( M \) near \( p \), that is, if \( V \) is given in some neighborhood of \( p \) by holomorphic functions \( f_1, \ldots, f_k \) with rank \( \mathcal{J}(f) = k \); the locus of smooth points of \( V \) is denoted \( V^* \). A point \( p \in V \setminus V^* \) is called a singular point of \( V \); the singular locus \( V \setminus V^* \) of \( V \) is denoted \( V_s \). \( V \) is called smooth or nonsingular if \( V = V^* \), i.e. if \( V \) is a submanifold of \( M \).

In particular, if \( p \) is a point of an analytic hypersurface \( V \subset M \) given in terms of local coordinates \( z \) by the function \( f \), we define the multiplicity \( \text{mult}_p(V) \) to be the order of vanishing of \( f \) at \( p \), that is, the great integer \( m \) such that all partial derivatives
\[
\frac{\partial^k f}{\partial z_{i_1} \cdots \partial z_{i_k}}(p) = 0, \quad k \leq m - 1.
\]

We should mention here a piece of terminology that is pervasive in algebraic geometry: the word generic. When we are dealing with a family of objects parametrized locally by a complex manifold or an analytic subvariety of a complex manifold, the statement that “a (or the) generic member of the family has a certain property” means exactly that “the set of objects in the family that do not have that property is contained in a subvariety of strictly smaller dimension.”

In general, it will be clear how the objects in our family are to be parametrized. One exception will be a reference to “the generic \( k \)-plane in \( \mathbb{P}^n \)”: until the section on Grassmannians, we have – at least officially – no way of parametrizing linear subspaces of projective space. The fastidious reader may substitute “the linear span of the generic \( (k + 1) \)-tuple of points in \( \mathbb{P}^{n+1} \)”.

**Proposition 3.** \( V_s \) is contained in an analytic subvariety of \( M \) not equal to \( V \).

**Proof.** For \( p \in V \) let \( k \) be the largest integer such that there exist \( k \) functions \( f_1, \ldots, f_k \) in a neighborhood \( U \) of \( p \) vanishing on \( V \) and such that \( \mathcal{J}(f) \) has \( k \times k \) min or not everywhere singular on \( V \); we may assume that \( |(\partial^j f_i/\partial z_j)|_{1 \leq i,j \leq k} \neq 0 \) on \( V \). Let \( U' \subset U \) be the locus of \( |(\partial^j f_i/\partial z_j)|_{1 \leq i,j \leq k} \neq 0 \) and \( V' \) the locus \( f_1 = \cdots = f_k = 0 \). Then \( V' = V \cap U' \) is a complex submanifold of \( U' \), and for any holomorphic function \( f \) vanishing on \( V' \) the differential \( df \equiv 0 \) on \( V' \), i.e. \( f \) is constant on \( V' \). It follows that for \( q \in V' \) near \( p \), \( V = V' \) is a manifold in a neighborhood of \( q \) and so \( V_s \subset \{ |(\partial^j f_i/\partial z_j)|_{1 \leq i,j \leq k} = 0 \} \).

It is in fact the case that \( V_s \) is an analytic subvariety on \( M \) – if we choose local defining functions \( f_1, \ldots, f_l \) for \( V \) carefully, \( V_s \) will be the common zero locus of the determinants of the \( k \times k \) minors of \( \mathcal{J}(f) \). For our purposes, however, we simply need to know that the singular locus of an analytic variety is comparatively small, and so we will not prove this stronger assertion.
Proposition 4. An analytic variety $V$ is irreducible if and only if $V^*$ is connected.

Proof. One direction is clear: if $V = V_1 \cup V_2$ with $V_1, V_2 \subsetneq V$ analytic varieties, then $(V_1 \cap V_2) \subset V$, so $V^*$ is disconnected.

The converse is harder to prove in general; since we will use it only for analytic hypersurfaces, we will prove it in this case. Suppose $V^*$ is disconnected, and let $\{V_i\}$ denote the connected components of $V^*$; we want to show that $\overline{V}_i$ is an analytic variety. Let $p \in \overline{V}_i$ be any point, $f$ a defining function for $V$ near $p$, and $z = (z_1, \ldots, z_n)$ local coordinates around $p$; we may assume that $f$ is a Weierstrass polynomial of degree $k$ in $z_n$. Write

$$g = \alpha \cdot f + \beta \cdot \frac{\partial f}{\partial z_n}, \quad g \neq 0 \in \mathcal{O}_n;$$

then for $\Delta$ some polydisc around $p$ and $\Delta'$ a polydisc in $\mathbb{C}^{n-1}$, the projection map $\pi: (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1})$ expresses $V_i \cap (\Delta \setminus (g = 0))$ as a covering space of $\Delta' \setminus (g = 0)$. Let $\{w_\nu(z')\}$ denote the $z_n$-coordinates of the points $\pi^{-1}(z')$ for $z' = (z_1, \ldots, z_{n-1}) \in \Delta' \setminus (g = 0)$ and let $\sigma_1(z'), \ldots, \sigma_k(z')$ denote the elementary symmetric functions of the $w_\nu$. The functions $\sigma_i$ are well-defined and bounded on $\Delta' \setminus (g = 0)$, and so extend to $\Delta'$; the function

$$f_i(z) = z_n^k + \sigma_1(z') z_n^{k-1} + \cdots + \sigma_k(z')$$

is thus holomorphic in a neighborhood of $p$ and vanishes exactly on $\overline{V}_i$.

Definition 11. We take the dimension of an irreducible analytic variety $V$ to be the dimension of the complex manifold $V^*$; we say that a general analytic variety is of dimension $k$ if all of its irreducible components are.

A note: if $V \subset M$ is an analytic subvariety of a complex manifold $M$, then we may define the tangent cone $T_p(V) \subset T'_p(M)$ to $V$ at any point $p \in V$ as follows: if $V = (f = 0)$ is an analytic hypersurface, and in terms of holomorphic coordinates $z = (z_1, \ldots, z_n)$ on $M$ centered around $p$ we write

$$f(z) = f_m(z) + f_2(z) + \cdots$$

with $f_k(z)$ a homogeneous polynomial of degree $k$ in $z$, then the tangent cone to $V$ at $p$ is taken to be the subvariety of $T'_p(M) = \mathbb{C}\{\partial/\partial z_i\}$ defined by

$$\left\{ \sum \alpha_i \frac{\partial}{\partial z_i} : f_m(\alpha_1, \ldots, \alpha_n) = 0 \right\}.$$

In general, then, the tangent cone to an analytic variety $V \subset M$ at $p \in V$ is taken to be the intersection of the tangent cones at $p$ to all local analytic hypersurfaces in $M$ containing $V$. In case $V$ is smooth at $p$, of course, this is just the tangent space to $V$ at $p$.

More geometrically, the tangent cone $T_p(V)$ may be realized as the union of the tangent lines at $p$ to all analytic arcs $\gamma: \Delta \to V \subset M$.

The multiplicity of a subvariety $V$ of dimension $k$ in $M$ at a point $p$, denoted $\text{mult}_p(V)$, is taken to be the number of sheets in the projection, in a small coordinate polydisc on $M$ around $p$, of $V$ onto a generic $k$-dimensional polydisc; note that $p$ is a smooth point of $V$ if and only if $\text{mult}_p(V) = 1$. In general, if $W \subset M$ is an irreducible subvariety, we define the multiplicity $\text{mult}_W(V)$ of $V$ along $W$ to be simply the multiplicity of $V$ at a generic point of $W$. 
REFERENCES